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Transport properties of a rectangular array of highly conducting cylinders

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Abstract. The method of functional equations is applied to evaluate the effective conductivity tensor for a rectangular array of highly conducting cylinders. The Rayleigh sum S_2 is calculated by Eisenstein and Weierstrass functions. Approximate analytical formula for the effective conductivity tensor are deduced.

Key words: effective transport properties, functional equation, Eisenstein functions, heat conduction, composite materials.

1. Introduction

Composite materials play an important role in many branches of engineering. Typically, in such materials, the physical parameters (such as electrical and heat conduction, elasticity coefficient,...) are discontinuous and vary between the different values characterizing each of the components. When these components are intimately mixed, these parameters vary very rapidly and the microscopic structure becomes complicated. On the other hand, we may expect to get a good approximation of the macroscopic behavior of such a heterogeneous material by a special kind of averaging of the properties of components. For engineers it is interesting to know formulae relating the macroscopic (effective) and microscopic properties, because it allows media to be created with unusual properties and the optimal design problem to be discussed. The results are of interest in thin films to calculate their optical properties, where films consisting of columns of one material in a matrix of another material are observed. In the field of materials physics two phase materials containing fibre inclusions often occur. Knowledge of their electrical or thermal conductivities is valuable in applied physics.

In the present paper we study the electrical properties of a rectangular array of cylinders. The same formalism and results are immediately applicable to many other problems governed by Laplace's equation; *e.g.* thermal conductivity, dielectric constant, permeability, modulus of torsion.

The problem of calculating the effective transport properties of a rectangular array of cylinders has received much attention. A number of workers have been inspired by a seminal paper of Lord Rayleigh [1]. McPhedran *et al.* [2–6] obtain an infinite set of linear algebraic equations for the multiple coefficients, which can be truncated to give various low-order formulae to calculate the effective conductivity tensor

$$\Lambda_e = \begin{pmatrix} \lambda_e^x & \lambda_e^{xy} \\ \lambda_e^{xy} & \lambda_e^y \end{pmatrix}$$

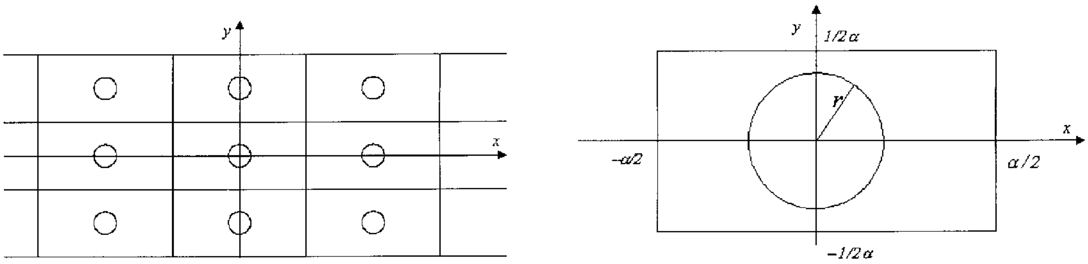

Figure 1. The rectangular array of cylinders and the unit cell.

of a square or hexagonal array of cylinders. Sangani and Yao [7] proposed an efficient method to calculate Λ_e for a square cell containing many cylinders whose size and location of the centers are arbitrary. The latter method is based on the same infinite system as in the method of Rayleigh. Bergman [8] derived an analytic representation of Λ_e related to the bounds on the coefficients of Λ_e. The latest results in this direction are represented by Bergman and Dunn [9] and Clark and Milton [10]. Using the method of collocation, Kolodziej [11–12] calculated numerically Λ_e of regular arrays of cylinders. Mityushev [13–19] applied the functional equation method to derive exact and approximate analytic formulae for arbitrary doubly periodic cell with arbitrary circular inclusions.

The case of highly conducting cylinders requires special attention. McPhedran *et al.* [5] calculated the square-array transport coefficient for arbitrary high cylinder conductivity and arbitrary small cylinder separations. Asymptotic formulae have been deduced, and their accuracy has been discussed.

In the present paper an asymptotic analysis of Mityushev's functional equation is applied to study the effective conductivity tensor Λ_e for a rectangular array of cylinders. The method of functional equations is extended to the case of highly conducting cylinders. This method allows us to consider rectangular arrays. It generalizes the results [2–7] devoted only to square and triangular arrays of cylinders. The final formulae for Λ_e are written in terms of the modified Eisenstein functions which are closely related to the elliptic functions. It is shown that such a representation leads to analytic asymptotic formulae with given arbitrary accuracy with respect to volume fraction. The formulae (26), (27) and (28) obtained in Section 5 are of practical interest, because many thin films with interesting electrical, mechanical and optical properties exhibit a columnar structure.

2. Functional equation

Consider a lattice Q defined by two fundamental translation vectors $\alpha > 0$ and $\mathrm{i}\alpha^{-1}$ in the complex plane $\mathbb{C}$. The zero cell Q_0, the basis of Q, is the rectangle $\{z = x+\mathrm{i}y = t_1\alpha + t_2\,\mathrm{i}\alpha^{-1}$, $-1/2 < t_j < 1/2,\ j = 1, 2\}$. The area of cell Q_0, $|Q_0| = 1$. Let $\{e_j\}_{j=0}^{\infty}$ be an ordered set of the complex numbers $m_1\alpha + m_2\,\mathrm{i}\alpha^{-1}$ arranged in accordance with the Eisenstein summation method (see Section 3). Here m_1 and m_2 are integers, $e_0 := 0$. The lattice $\mathcal{Q}$ consists of the cells $Q_j = Q_0 + e_j := \{z \in \mathbb{C}, z - e_j \in Q_0\}$.

Consider the disk $D_1 := \{z \in \mathbb{C}, |z| < r\}$ in the zero cell Q_0. Let $D := Q_0 - \overline{D_1}$. We study the conductivity of the doubly periodic composite material, when the domains $D + e_j$ and $D_1 + e_j$ are occupied by materials of unit and $\lambda_1 > 0$ conductivity, respectively (Figure 1).

The potentials $u(z)$ and $u_1(z)$ are harmonic in $D + e_j$ and $D_1 + e_j$ $(j = 0, 1, 2, \ldots)$ with the boundary conditions

$$u = u_1, \qquad \frac{\partial u}{\partial n} = \lambda_1 \frac{\partial u_1}{\partial n} \quad \text{on } L, \tag{1}$$

where $\partial/\partial n$ is the normal derivative, $L := \{z \in \mathbb{C}, |z| = r\}$. The external field is applied in the x-direction

$$u(z + \alpha) = u(z) + \alpha, \qquad u(z + \mathrm{i}\alpha^{-1}) = u(z). \tag{2}$$

If the conductivity of the inclusions $D_1 + e_j$ tends to infinity, then condition (1) becomes

$$u = u_1 \quad \text{on } L, \tag{3}$$

where u_1 is a constant. It is problem defined by (3), (2) that is discussed in the present paper. It is convenient for us to start by considering problem (1), (2) with finite λ_1. Then it is assumed that $\lambda_1 \to \infty$.

Following Mityushev [15, 19] we reduce problem (1), (2) to a functional equation. We use the normal and tangent derivatives on the curve L

$$\frac{\partial}{\partial n} = \cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y}; \qquad \frac{\partial}{\partial s} = -\sin\theta \frac{\partial}{\partial x} + \cos\theta \frac{\partial}{\partial y},$$

where $\mathbf{n} = (\cos\theta, \sin\theta)$ is the normal vector to L. Applying the operator $\partial/\partial s$ to the first relation (1), we obtain

$$-\sin\theta \frac{\partial u}{\partial x} + \cos\theta \frac{\partial u}{\partial y} = -\sin\theta \frac{\partial u_1}{\partial x} + \cos\theta \frac{\partial u_1}{\partial y}. \tag{4}$$

The second relation (1) can be written in the form

$$\cos\theta \frac{\partial u}{\partial x} + \sin\theta \frac{\partial u}{\partial y} = \lambda_1 \cos\theta \frac{\partial u_1}{\partial x} + \lambda_1 \sin\theta \frac{\partial u_1}{\partial y}. \tag{5}$$

Let us introduce the complex potentials

$$\psi(z) := \frac{\lambda_1 + 1}{2} \left(\frac{\partial u_1}{\partial x} - \mathrm{i} \frac{\partial u_1}{\partial y} \right) \quad \text{and} \quad \phi(z) := \frac{\partial u}{\partial x} - \mathrm{i} \frac{\partial u}{\partial y}$$

which are analytic in D_1 and D, respectively, and continuous in the closures of the domains considered. Substituting

$$\frac{\partial u_1}{\partial x} = \frac{1}{\lambda_1 + 1}(\psi + \bar{\psi}), \qquad \frac{\partial u_1}{\partial y} = \frac{\mathrm{i}}{\lambda_1 + 1}(\psi - \bar{\psi}),$$

$$\frac{\partial u}{\partial x} = \tfrac{1}{2}(\phi + \bar{\phi}), \qquad \frac{\partial u}{\partial y} = \frac{\mathrm{i}}{2}(\phi - \bar{\phi})$$

in (4), (5) we obtain the following $\mathbb{R}$-linear problem [15]

$$\phi(t) = \psi(t) + \rho \overline{n^2 \psi(t)}, \quad t \in L$$

with respect to the functions $\phi(z)$ and $\psi(z)$ which are analytic in D_1 and D, respectively. Here $\rho := (\lambda_1 - 1)(\lambda_1 + 1)^{-1}$ is a Bergman's parameter [8–9], the normal vector $\mathbf{n}$ is represented as a complex number $n := \cos\theta + \mathrm{i}\sin\theta$. We denote the position inside domains by the complex variable $z = x + \mathrm{i}y$; the position along L – by the complex variable $t = r\,\mathrm{e}^{\mathrm{i}\theta}$. Calculating $\overline{n^2} = (\cos\theta - \mathrm{i}\sin\theta)^2 = (r/t)^2$ we arrive at the relation

$$\phi(t) = \psi(t) + \rho\left(\frac{r}{t}\right)^2\overline{\psi(t)}, \quad t \in L. \tag{6}$$

Equalities (2) imply that the function $\phi(z)$ is doubly periodic in $\mathbb{C}$.

Following [15–16, 19] we introduce the function

$$\sum_{j=1}^{\infty}(z - e_j)^{-2} := \sum_{j=1}^{\infty}[(z - e_j)^{-2} - e_j^{-2}] + S_2 = \mathcal{P}(z) - z^{-2} + S_2, \tag{7}$$

and the series

$$\sum_{j=1}^{\infty}(z - e_j)^{-2}\overline{\psi\left(\frac{r^2}{\overline{z - e_j}}\right)}$$

$$:= \sum_{j=1}^{\infty}(z - e_j)^{-2}\left[\overline{\psi\left(\frac{r^2}{\overline{z - e_j}}\right)} - \overline{\psi(0)}\right] + \overline{\psi(0)}\sum_{j=1}^{\infty}(z - e_j)^{-2},$$

where

$$\sum_{j=1}^{\infty}e_j^{-2} = S_2 := \frac{\pi^2}{\alpha^2}\left(\frac{1}{3} - 2\sum_{m=1}^{\infty}\sinh^{-2}(\pi m\alpha^{-2})\right), \tag{8}$$

$\mathcal{P}(z)$ is the Weierstrass function [21]. Formula (8) is discussed in the next section.

Introduce the function

$$\Phi(z) := \begin{cases} \psi(z) - \rho r^2\displaystyle\sum_{j=1}^{\infty}(z - e_j)^{-2}\overline{\psi\left(\frac{r^2}{\overline{z - e_j}}\right)}, & |z| \leqslant r, \\ \phi(z) - \rho r^2\displaystyle\sum_{j=0}^{\infty}(z - e_j)^{-2}\overline{\psi\left(\frac{r^2}{\overline{z - e_j}}\right)}, & z \in D. \end{cases}$$

Using (6) and applying the principles of analytic function theory, Mityushev [15] has proved that $\Phi(z) \equiv 1$. This relation yields the following functional equation

$$\psi(z) = \rho r^2\sum_{j=1}^{\infty}(z - e_j)^{-2}\overline{\psi\left(\frac{r^2}{\overline{z - e_j}}\right)} + 1, \quad |z| \leqslant r, \tag{9}$$

with respect to the function $\psi(z)$ analytic in $|z| < r$ and continuous in $|z| \leqslant r$.

We now proceed to calculate the component

$$\lambda_e^x = \iint_D \frac{\partial u}{\partial x}\,\mathrm{d}x\,\mathrm{d}y + \lambda_1\iint_{D_1}\frac{\partial u_1}{\partial x}\,\mathrm{d}x\,\mathrm{d}y$$

of the effective conductivity tensor Λ_e. Using Green's formula we have

$$\iint_D \frac{\partial u}{\partial x}\,\mathrm{d}x\,\mathrm{d}y = \int_{\partial Q_0} u\,\mathrm{d}y - \int_L u\,\mathrm{d}y. \tag{10}$$

Using (2) we have

$$\int_{\partial Q_0} u\,\mathrm{d}y = \int_{-(1/2\alpha)}^{1/2\alpha} \left[u\left(-\frac{\alpha}{2}+\mathrm{i}y\right) - u\left(\frac{\alpha}{2}+\mathrm{i}y\right)\right]\mathrm{d}y = 1.$$

The integral $\int_L u\,\mathrm{d}y$ is calculated by the mean value theorem of harmonic functions in a disk. We have

$$\int_L u\,\mathrm{d}y = \int_L u_1\,\mathrm{d}y = v\frac{\partial u_1}{\partial x}(0),$$

where $v = \pi r^2$ is the area fraction of the inclusions. Hence, λ_e^x takes the form

$$\lambda_e^x = 1 + (\lambda_1 - 1)v\frac{\partial u_1}{\partial x}(0).$$

Similar arguments applied to

$$\lambda_e^{xy} = \iint_D \frac{\partial u}{\partial y}\,\mathrm{d}x\,\mathrm{d}y + \lambda_1 \iint_{D_1} \frac{\partial u_1}{\partial y}\,\mathrm{d}x\,\mathrm{d}y$$

yield the relation

$$\lambda_e^{xy} = (\lambda_1 - 1)v\frac{\partial u_1}{\partial y}(0). \tag{11}$$

Two real relations (10) and (11) imply one complex equality

$$\lambda_e^x - \mathrm{i}\lambda_e^{xy} = 1 + 2\rho v\psi(0).$$

We note that symmetry of the problem implies $\lambda_e^{xy} = 0$. Hence, $\psi(0)$ is real and

$$\lambda_e^x = 1 + 2\rho v\psi(0). \tag{12}$$

If the conductivity of cylinders λ_1 is much greater than the conductivity of the host 1, then one can assume that $\lambda_1 = +\infty$, and hence $\rho = 1$. The functional equation (9) then becomes

$$\psi(z) = r^2 \sum_{j=1}^{\infty} (z - e_j)^{-2} \overline{\psi\left(\frac{r^2}{\overline{z - e_j}}\right)} + 1, \quad |z| \leqslant r. \tag{13}$$

Convergence of the method of successive approximations for Equation (13) in the case $|\rho| = 1$ has not been previously investigated. In the present paper this convergence question is also not addressed. We propose only a simple algorithm to get approximate analytic formulae for $\psi(z)$ and λ_e^x with arbitrary given accuracy.

3. Elliptic functions according to Eisenstein

In order to calculate the effective transport conductivity tensor Rayleigh [1] introduced a conditionally convergent sum

$$S_2 = \sum_{j=1}^{\infty} e_j^{-2}, \tag{14}$$

defined in the following way

$$S_2 := \lim_{N\to\infty} \left(\sum_{m_1=-N}^{N} \lim_{M\to\infty} \sum_{m_2=-M}^{M} (m_1\alpha + \mathrm{i} m_2 \alpha^{-1})^{-2} \right) \tag{15}$$

and calculated by (8). The method of summation in (14) is determined by (15). The sum S_2 in (15) was introduced by Eisenstein. A modern survey of the Eisenstein approach is due to Weil [20]. In the present section Eisenstein functions and their relation to the Rayleigh sums are discussed.

Eisenstein introduced the following functions

$$E_n(z) := \sum_j (z + e_j)^{-n}. \tag{16}$$

If $n \geqslant 3$, then the series (16) is absolutely convergent. If $n = 1$ or $n = 2$ Eisenstein defined these series as (15) by the following summation method

$$\sum_e := \lim_{N\to\infty} \left(\sum_{m_1=-N}^{N} \lim_{M\to\infty} \sum_{m_2=-M}^{M} \right).$$

The function $E_2(z)$ can be represented by the absolutely convergent series

$$E_2(z) = \left(\frac{\pi}{\alpha}\right)^2 \left[\sin^{-2}(\pi z \alpha^{-1}) + 2 \sum_{m=-\infty}^{\infty} \sin^{-2}(\pi \alpha^{-1}(z + \mathrm{i} m \alpha^{-1})) \right]. \tag{17}$$

The Eisenstein function $E_2(z)$ is related to the Weierstrass function $\wp(z)$ by the formula

$$E_2(z) - S_2 = \wp(z),$$

where S_2 has the form (15). This formula implies the equality

$$S_2 = (E_2(z) - \wp(z))_{z=0} = (E_2(z) - z^{-2})_{z=0},$$

since $(\wp(z) - z^{-2})_{z=0} = 0$. Using (17) we arrive at formula (8).

The Eisenstein function $E_1(z) = \sum_e (z + e_j)^{-1}$ is related to the Weierstrass function $\zeta(z)$ by the formula

$$\zeta(z) = E_1(z) + S_2 z.$$

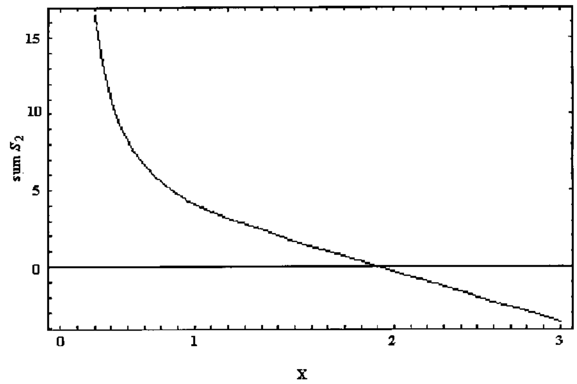

Figure 2. The Rayleigh sum $S_2(X)$ calculated with (18).

Using the relations

$$\zeta(z+\alpha)-\zeta(z)=2\zeta(\alpha/2), \qquad E_1(z+\alpha)-E_1(z)=0,$$

we obtain the fundamental equality

$$S_2=\alpha^{-1}2\zeta(\alpha/2).$$

It follows from elliptic function theory [21] that

$$S_2=\frac{\pi^2}{\alpha^2}\left(\frac{1}{3}-8\sum_{m=1}^{\infty}\frac{mh^{2m}}{1-h^{2m}}\right), \tag{18}$$

where $h=\exp(-\pi\alpha^{-2})$. So we have proved that the formulae (8), (15) and (18) must give the same result. The formula (18) is very effective in calculation, because $h \leqslant \exp(-\pi) \approx 0{\cdot}043$ for $0<\alpha \leqslant 1$. Mityushev [18] proved the identity

$$S_2(\alpha^2)+S_2(\alpha^{-2})=2\pi, \tag{19}$$

S_2 is considered as a function on α^2. The formula (19) allows us to calculate S_2, $\alpha>1$. The function $S_2(X)$ is represented in Figure 2.

Introduce the modified Eisenstein functions

$$\sigma_n(z):=E_n(z)-z^{-n}, \quad n=1,2,\ldots \tag{20}$$

with their Taylor expansions

$$\sigma_n(z)=(-1)^n\sum_{l=0}^{\infty}\binom{l+n-1}{n-1}S_{l+n}z^l. \tag{21}$$

Here S_2 is defined by (15) and calculated by (8) or (18). The sums

$$S_n:=\sum_{l=0}^{\infty}e_j^{-n}, \quad n=3,4,\ldots$$

are absolutely convergent. It is known [21] that $S_n = 0$ for odd n and S_n are real for even n. The sums S_n take real values in general only for rectangular arrays of cylinders.

4. Method of Rayleigh

The method of Rayleigh [1] has been discussed in [2–20] and others. Starting from Equation (13) we can obtain an infinite system of linear algebraic equations following this approach. We look for a function $\psi(z)$ from (13) in the form of the Taylor expansion

$$\psi(z) = \sum_{m=0}^{\infty} \alpha_m z^m.$$

Substituting this expansion in (13) we obtain

$$\sum_{m=0}^{\infty} \alpha_m z^m = \sum_{m=0}^{\infty} \overline{\alpha_m} r^{2(m+1)} \sigma_{m+2}(z) + 1, \quad |z| \leqslant r, \tag{22}$$

where $\sigma_n(z)$ are the modified Eisenstein functions (20). Substituting (21) in (22) we obtain the infinite $\mathbb{R}$-linear algebraic system

$$\alpha_l = \sum_{m=0}^{\infty} (-1)^m \binom{l+m+1}{m+1} S_{l+m+2} r^{2(m+1)} \overline{\alpha_m} + \delta_{l0}, \quad l = 0, 1, 2, \ldots \tag{23}$$

with respect to α_l. Here δ_{l0} is the Kronecker symbol. Calculating the real part of (23), we arrive at the Rayleigh system.

We do not use the system (23) to calculate $\psi(z)$ and the effective conductivity tensor. We note only that the coefficients α_l $(l = 0, 1, 2, \ldots)$ satisfying (23) are analytic functions on r^2. Therefore, the function $\psi(z) = \psi(z, r^2)$ is analytic in r^2.

5. Solution to Equation (13)

It follows from the previous section that we can look for $\psi(z)$ in the form

$$\psi(z) = \psi(z, r^2) = \sum_{m=0}^{\infty} \psi_m(z) r^{2m}. \tag{24}$$

Substitute (24) in (13) and select the terms with r^{2m} $(m = 0, 1, 2, \ldots)$. In the results we have the following scheme of successive approximations to calculate $\psi_m(z)$

$$\psi_0(z) = 1, \qquad \psi_m(z) = \sum_{l+s=m-1} \overline{\psi_{ls}} \sigma_{s+2}(z), \quad m = 1, 2, \ldots, \tag{25}$$

where $\psi_l(z) = \sum_{s=0}^{\infty} \psi_{ls} z^s$. The finite sum $\sum_{l+s=m-1}$ contains the terms in $l = m-1$, $s = 0$; $l = m-3$, $s = 2$; $\ldots$; $l = 0$, $s = m-1$. It follows from (21) that $\sigma_n(0) = (-1)^n\ S_n = 0$ for odd n, and $\sigma_n(0) = S_n$ for even n. Therefore, $\psi_{ls} = 0$ for odd s.

We calculate

$$
\begin{aligned}
\psi_1(z) &= \overline{\psi_{00}}\sigma_2(z) = \sigma_2(z) = \sum_{l=0}^{\infty}(l+1)S_{l+2}z^l,\\
\psi_2(z) &= \overline{\psi_{10}}\sigma_2(z) = \sigma_2(0)\sigma_2(z) = S_2\sigma_2(z),\\
\psi_3(z) &= \overline{\psi_{20}}\sigma_2(z) + \overline{\psi_{02}}\sigma_4(z) = S_2^2\sigma_2(z),\\
\psi_4(z) &= \overline{\psi_{30}}\sigma_2(z) + \overline{\psi_{12}}\sigma_4(z) = S_2^3\sigma_2(z) + 3S_4\sigma_4(z),\\
\psi_5(z) &= \overline{\psi_{40}}\sigma_2(z) + \overline{\psi_{22}}\sigma_4(z) = (S_2^4 + 3S_4^2)\sigma_2(z) + 3S_2S_4\sigma_4(z),\\
\psi_6(z) &= \overline{\psi_{50}}\sigma_2(z) + \overline{\psi_{32}}\sigma_4(z) + \overline{\psi_{14}}\sigma_6(z)\\
&= (S_2^5 + 6S_2S_4^2)\sigma_2(z) + 3S_2^2S_4\sigma_4(z) + 5S_6\sigma_6(z),\\
\psi_7(z) &= \overline{\psi_{60}}\sigma_2(z) + \overline{\psi_{42}}\sigma_4(z) + \overline{\psi_{24}}\sigma_6(z)\\
&= (S_2^6 + 9S_2^2S_4^2 + 5S_6^2)\sigma_2(z) + 3S_4(S_2^3 + 10S_6)\sigma_4(z) + 5S_2S_6\sigma_6(z),\\
\psi_8(z) &= \overline{\psi_{70}}\sigma_2(z) + \overline{\psi_{52}}\sigma_4(z) + \overline{\psi_{34}}\sigma_6(z) + \overline{\psi_{16}}\sigma_8(z)\\
&= (S_2^7 + 12S_2^3S_4^2 + 10S_2S_6^2 + 30S_4^2S_6)\sigma_2(z)\\
&\quad +3S_4(S_2^4 + 3S_4^2 + 10S_2S_6)\sigma_4(z) + 5S_2^2S_6\sigma_6(z) + 7S_8\sigma(z).
\end{aligned}
$$

Using (25) we can calculate the next functions $\psi_{2n}(z)$.
These equalities and (12) imply

$$
\begin{aligned}
\lambda_e^x &= 1 + 2v\sum_{m=0}^{\infty}\psi_m(0)\pi^{-m}v^m = \frac{1+v(2-S_2/\pi)}{1-vS_2/\pi} + \frac{6S_4^2\pi^{-4}v^5}{(1-vS_2/\pi)^2}\\
&\quad +10S_2S_6(2+3vS_2/\pi)\pi^{-7}v^7 + 60S_4^2S_6(1+2vS_2/\pi)\pi^{-7}v^7\\
&\quad +2(9S_4^4 + 7S_8^2)\pi^{-8}v^9 + O(v^{10}), \quad \text{as } v \to 0. \qquad (26)
\end{aligned}
$$

Let us consider the components λ_e^x and λ_e^y of the effective conductivity tensor Λ_e as functions on α^2. Then

$$\lambda_e^y(\alpha^2) = \lambda_e^x(\alpha^{-2}).$$

This formula allows us to calculate $\lambda_e^y = \lambda_e^y(\alpha^2)$, using (26). If we change α^2 into α^{-2}, then $S_2(\alpha^{-2}) = 2\pi - S_2(\alpha^2)$ in accordance with (19). The lattice sums S_{2n} ($n \geqslant 2$) do not change. Therefore,

$$
\begin{aligned}
\lambda_e^y &= \frac{1+vS_2/\pi}{1-v(2-S_2/\pi)} + \frac{6S_4^2\pi^{-4}v^5}{(1-v(2-S_2/\pi))^2}\\
&\quad +10S_2S_6(2+3v(2-S_2/\pi))\pi^{-7}v^7\\
&\quad +60S_4^2S_6(1+2v(2-S_2/\pi))\pi^{-7}v^7\\
&\quad +2(9S_4^4 + 7S_8^2)\pi^{-8}v^9 + O(v^{10}), \quad \text{as } v \to 0. \qquad (27)
\end{aligned}
$$

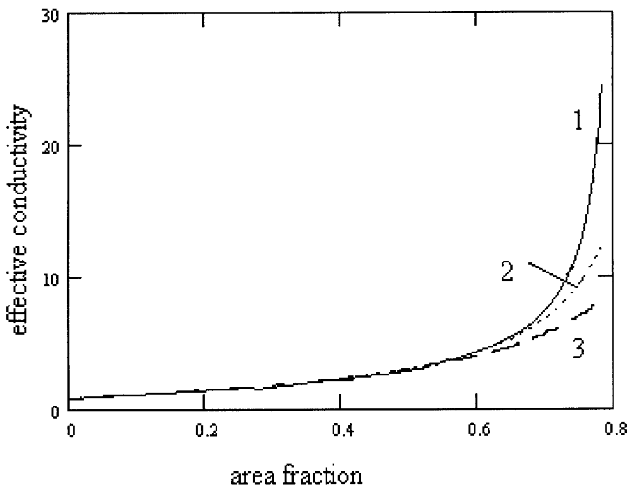

Figure 3. The effective conductivity coefficient of a square array of cylinders: curve 1 calculated with (30); curve 2 calculated with (28) and curve 3 calculated with (29).

The tensor Λ_e has the form

$$\Lambda_e = \begin{pmatrix} \lambda_e^x & 0 \\ 0 & \lambda_e^y \end{pmatrix},$$

where λ_e^x and λ_e^y are calculated with (26) and (27).

For a square array of cylinders the effective conductivity is a scalar value $\lambda_e = \lambda_e^x = \lambda_e^y$, and $S_2 = \pi$, $S_4 \approx 3{\cdot}1512112$, $S_6 = 0$, $S_8 \approx 4{\cdot}2557732$. Then (26) and (27) yield

$$\lambda_e = \frac{1+v}{1-v} + 6S_4^2\pi^{-4}\frac{v^5}{(1-v)^2} + 2(9S_4^2 + 7S_8^2)\pi^{-8}v^9 + O(v^{10}), \quad \text{as } v \to 0. \tag{28}$$

The first term in (28) corresponds to the well-known Clausius–Mossotti approximation

$$\lambda_e^x \approx \frac{1+v}{1-v}. \tag{29}$$

We compare formulae (28), (29) with the formula

$$\lambda_e \approx 1 + 2\rho v \Bigg/ \left(1 - \rho v - \frac{0{\cdot}305827 v^4}{\rho^{-2} - 1{\cdot}402958 v^8} - 0{\cdot}013362\rho^2 v^8\right) \tag{30}$$

for $\rho = 1$ from [6] in Figure 3. It is hard to say which formula is better. We know only the lower bound $\lambda_e \geqslant (1+v)/(1-v)$ for perfectly conducting cylinders. Both formulae (28) and (30) satisfy this inequality and contain the terms up to $O(v^9)$. Anyway, one can calculate the next terms $\psi_m(z)$ $(m > 9)$ by (25) and improve (28).

6. Concluding remarks

The Rayleigh sum S_2 is calculated by Eisenstein and Weierstrass functions for the rectangular array of cylinders. So it is justified that all different previous definitions of the Rayleigh sum of second order give the same result.

The method of functional equations is extended to the case of highly conducting cylinders. This allows us to deduce a simple iterative scheme (25) and an approximate analytical formulae (26) and (27) for the components of the effective conductivity tensor. We note that formulae (26) and (27) are valid for arbitrary rectangular arrays, formula (30) from [6] has been deduced only for a square array.

It is easily seen that the maximum area fraction of the inclusions $v_{\max}$ depends on α by the rule

$$v_{\max} = \frac{\pi}{4} \min(\alpha^2, \alpha^{-2}). \tag{31}$$

This corresponds to touching cylinders. Formula (28) can also be applied to $v = v_{\max}$, but for α separated from 1. In the case when v is close to $v_{\max}$ and $\alpha = 1$ we cannot use an expansion of λ_e^x in a neighborhood of the point $v = 0$ [4, 5]. Hence, formula (28) cannot be applied in this case.

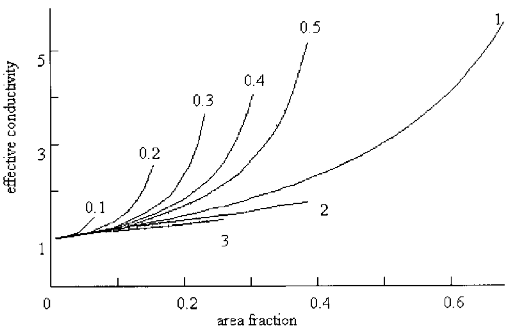

Figure 4. The coefficient $\lambda_e^x = \lambda_e^x(\alpha^2, v)$, where $\alpha = 0{\cdot}1, 0{\cdot}2, \ldots, 3$, the area fraction v changes along x-axes.

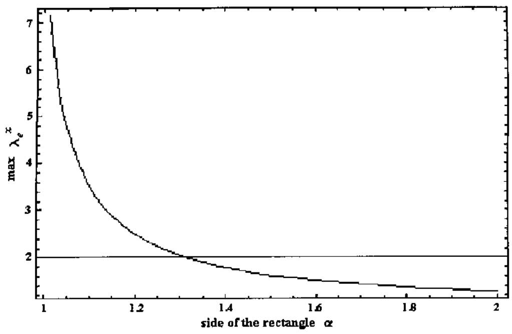

Figure 5. The coefficient $\lambda_e^x = \lambda_e^x(\alpha^2, v_{\max})$, where $v_{\max}$ is calculated with (31).

Figure 4 presents λ_e^x as a function on v with different fixed α. Here the restrictions (31) are given in account. The curves in Figure 4 end in the points (31) and show the maximum possible value λ_e^x for fixed α. Of course the maximum λ_e^x tends to infinity if α tends to 1. See McPhedran *et al.* [4–5]. Figure 5 presents the maximum possible λ_e^x as a function α with $v_{\max}$ calculated by (31).

Formulae (26) and (27) are easy to calculate and can be simply used in applied physics to evaluate macroscopic properties of thin films and materials containing fibre inclusions. See McPhedran *et al.* [2–6].

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References

1. Lord Rayleigh, On the influence of obstacles arranged in rectangular order upon the properties of a medium. *Phil. Mag.* 34 (1892) 481–502.
2. R. C. McPhedran and G. Milton, Bounds and exact theories for the transport properties of inhomogeneous media. *Appl. Phys.* A26 (1981) 207–220.
3. R. C. McPhedran, Transport properties of cylinder pairs and the square array of cylinders. *Proc. R. Soc. London* A408 (1986) 31–43.
4. R. C. McPhedran and G. Milton, Transport properties of touching cylinder pairs and of the square array of touching cylinders. *Proc. R. Soc. London* A411 (1987) 313–326.
5. R. C. McPhedran, L. Poladian and G. Milton, Asymptotic studies of closely spaced, highly conducting cylinders. *Proc. R. Soc. London* A415 (1988) 185–196.
6. W. T. Perrins, D. R. McKenzie and R. C. McPhedran, Transport properties of regular array of cylinders. *Proc. R. Soc. London* A369 (1979) 207–225.
7. A. S. Sangani and C. Yao, Transport properties in random arrays of cylinders. 1. Thermal conduction. *Phys. Fluids* 31 (1988) 2426–2434.
8. D. J. Bergman, The dielectric constant of a composite material – a problem in a classical physics. *Phys. Reports* C43 (1983) 378–407.
9. D. J. Bergman and K. J. Dunn, Bulk effective dielectric constant of a composite with a periodic microgeometry. *Phys. Rev.* B45 (1992) 13262–13271.
10. K. E. Clark and G. W. Milton, Optimal bounds correlating electric, magnetic and thermal properties of two-phase, two-dimensional composites. *Proc. R. Soc. London* A448 (1995) 161–190.
11. J. Kolodziej, Calculation of the effective thermal conductivity of a composite with parallel cylinders by the method of collocation. *Mech. Teoret. i Stosowana* 23 (1985) 355–373 (in Polish).
12. J. Kolodziej, Calculation of the effective thermal conductivity of a unidirectional composites. *Arch. Termodynamiki* 8 (1987) 101–107 (in Polish).
13. V. V. Mityushev, Steady heat conduction of two-dimensional composites. *Proc. IX Symp. Heat Mass Transfer, Augustow.* 2 (1995) 93–100.
14. V. V. Mityushev, Transport properties of double-periodic arrays of circular cylinders. *ZAMM* 77 (1997) 115–120.
15. V. V. Mityushev, Functional equations in a class of analytic functions and composite materials. *Demonstratio Mathematica* 30 (1997) 63–70.
16. V. V. Mityushev, Rayleigh's integral and the square array of cylinders. *Arch. Mech.* 47 (1995) 27–37.
17. V. V. Mityushev, A method of functional equations for boundary value problems of continuous media. *Reports Math. Phys.* 33 (1993) 137–147.
18. V. V. Mityushev, *Application of Functional Equation to Determination of the Effective Thermal Conductivity of Composite Materials.* Slupsk: WSP (1996) 155 pp (in Polish).
19. V. V. Mityushev, Finite and infinite composite materials and Rayleigh's sum. *Arch. Mech.* 49 (1997) 345–358.
20. A. Weil, *Elliptic Functions According to Eisenstein and Kronecker.* Berlin: Springer-Verlag (1976) 112 pp.
21. A. Hurwitz and R. Courant, *Allgemeine Funktionentheorie und Elliptische Funktionen.* Berlin: Springer-Verlag (1964) 274 pp.